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Induced Characters and Symplectic Sections

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1. INTRODUCTION

Let G be a finite group and $\chi \in \operatorname{Irr} G$ (the set of irreducible characters of G). Let N be a normal subgroup of G. I. M. Isaacs noticed that when G/N is supersolvable, then there is a subgroup H of G, and $\theta \in \operatorname{Irr} H$ such that $\theta^G = \chi$, while $\theta \mid N \in \operatorname{Irr} N$. This situation often can be exploited in inductive arguments (e.g., Step 2 in the proof of Theorem 3.2). In this paper we explore some aspects of such a configuration.

Let G be a solvable group with $N \lhd G$. When G/N is supersolvable, Dornhoff [1] found that restricting the Sylow subgroups of N to the modular or quaternion-free condition is the most general condition of the Sylow subgroups of N which guarantees that G is an M-group. When N satisfies the Dornhoff condition, we show in Theorem 4.1 the most general condition of G/N such that G is an M-group.

The results of Section 5 have been applied by Schacher and Seitz [9].

The notation in this paper is largely that of Huppert [2]. If H is a normal subgroup of the finite group G, and $\theta \in \operatorname{Irr} H$, then the inertia group of θ in G will be denoted $\mathbf{I}_G(\theta)$.

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2. PRELIMINARY RESULTS ABOUT CHARACTERS

In addition to the fundamental Frobenius Reciprocity Theorem and Clifford's theorem, the following lemmas will be useful.

LEMMA 2.1 ([4, Lemma 3.3]). *Let G be a finite group with subgroups A*

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and B such that $G = AB$ and let θ be any class function on A. Then $\theta^G \mid B = (\theta \mid (A \cap B))^B$.

The next lemma analyzes a situation which occurs frequently in this paper.

LEMMA 2.2. *Let $N \lhd G$, and $\chi \in \operatorname{Irr} G$. Suppose that $\chi \mid N \in \operatorname{Irr} N$. Suppose there is an $H \leqslant G$ and $\theta \in \operatorname{Irr} H$ with $\theta^G = \chi$. Then:*

(a) $G = HN$.

(b) $\theta \mid (H \cap N) \in \operatorname{Irr}(H \cap N)$.

(c) $\{\theta \mid (H \cap N)\}^N = \chi \mid N$.

Proof. Now $(\theta^{HN})^G = \theta^G = \chi$, so $\theta^{HN} \in \operatorname{Irr} HN$. By Frobenius reciprocity, θ^{HN} is a constituent of $\chi \mid HN$. But $\chi \mid N \in \operatorname{Irr} N$, so $\chi \mid HN \in \operatorname{Irr} HN$. Thus, $\theta^{HN} = \chi \mid HN$. But then $\theta^{HN}(1) = \chi(1) = \theta^G(1)$. Therefore, $HN = G$. By Lemma 2.1, $\chi \mid N = \theta^G \mid N = \{\theta \mid (H \cap N)\}^N$. Then $\{\theta \mid (H \cap N)\}^N \in \operatorname{Irr} N$, so $\theta \mid (H \cap N)$ must be an irreducible character of $H \cap N$.

The next result is useful sharpening of Clifford's theorem [3, Proposition 3].

THEOREM 2.3. *Let G be a finite group, H/K be an elementary abelian chief section of G. Let $\chi \in \operatorname{Irr} H$ and suppose χ is invariant in G. Then one of the following occurs.*

(a) $\chi \mid K \in \operatorname{Irr} K$.

(b) $\chi \mid K = e\theta$, *where* $\theta \in \operatorname{Irr} K$ *and* $e^2 = (H : K)$.

(c) $\chi \mid K = \theta_1 + \cdots + \theta_{(H:K)}$, *and the* $\theta_i \in \operatorname{Irr} K$ *are distinct.*

THEOREM 2.4. *Let G be a finite group, H/K an elementary abelian chief section of G. Let $\theta \in \operatorname{Irr} K$, and suppose that $G = \mathbf{I}_G(\theta)H$. Then one of the following occurs.*

(a) $\theta^H = \chi_1 + \cdots + \chi_{(H:K)}$, *and the* $\chi_i \in \operatorname{Irr} G$ *are distinct.*

(b) $\theta^H = e\chi$, *with* $\chi \in \operatorname{Irr} H$ *and* $e^2 = (H : K)$.

(c) $\theta^H \in \operatorname{Irr} H$.

3. INDUCING CHARACTERS OVER A NORMAL SUBGROUP

This section presents the fundamental definition for the paper, and some consequences.

DEFINITION. Let χ be an irreducible character of the finite group G, and N be a normal subgroup of G. Call χ *induced over N* if there is a subgroup H of G containing N and irreducible character θ of H so that $\theta^G = \chi$ while $\theta \mid N$ is irreducible. Such an H and θ will be said to induce χ over N.

Remarks. (1) I. M. Isaacs suggested the study of characters induced over a normal subgroup after he observed that if G/N is supersolvable, then every irreducible character of G is induced over N. Theorem 3.3 is a slight generalization of this observation.

(2) When G/N is abelian, R. L. Roth has also noted that every irreducible character of G is induced over N [8, Theorem 3.1].

(3) The irreducible character χ of G is monomial if and only if χ is induced over the identity subgroup of G. So the study of characters induced over a normal subgroup is a generalization of the study of monomial characters.

LEMMA 3.1. *Let χ be an irreducible character of the finite grqup G which is induced over the normal subgroup N of G. Let $K \lhd G$ be a subgroup of* Ker χ. *Then χ (as a character of G/K) is induced over NK/K.*

The corollary to the next two theorems extends a result of Dornhoff [1].

THEOREM 3.2. *Let G be a finite group, $N \lhd G$. Suppose N is solvable, the odd Sylow subgroups of N are modular, and the Sylow 2-subgroups of N are quaternion-free. Let $\chi \in \operatorname{Irr} G$ and suppose that χ is induced over N. Then χ is a monomial character of G.*

Proof. By induction on $|G|$.

Step 1. *χ is faithful.* Let $K = \operatorname{Ker} \chi$. By Lemma 3.1, χ, as a character of G/K, is induced over NK/K. Now $NK/K \simeq N/(N \cap K)$ inherits the hypothesis on N, so, if $K > 1$, then by induction χ is a monomial character of G/K and hence it is a monomial character of G.

Step 2. *$\chi \mid N$ is irreducible.* Let $H \leqslant G$ with $\theta \in \operatorname{Irr} H$ induce χ over N. Trivially, θ, as a character of H, is induced by θ over N. If $H < G$, then by induction, θ is a monomial character of H, so $\chi = \theta^G$ is a monomial character of G.

Step 3. *χ is primitive.* Let $H \leqslant G$, and suppose, on the contrary, that there is $\theta \in \operatorname{Irr} H$ with $\theta^G = \chi$. Let $L = H \cap N$. Then $L \lhd H$, and since $L \leqslant N$, the assumptions on N are inherited by L. By Lemma 2.2(b), $\theta \mid L \in \operatorname{Irr} L$. Thus, θ as a character of H, is induced over L. If $H < G$, then by induction θ is a monomial character of H, so $\chi = \theta^G$ is a monomial character of G.

Step 4. *Let $A \lhd G$, with A abelian. Then $A \leqslant \mathbf{Z}(G)$, and A is cyclic.* By Step 3 and Clifford's theorem, $\chi \mid A = \chi(1)\lambda$, for some linear character λ of A. Since χ is faithful, $A \leqslant \mathbf{Z}(G)$, and since A is represented by scalar matrices in a representation of G affording χ, A is cyclic.

Step 5. *N is nilpotent.* Let K be the smallest normal subgroup of N for which N/K is nilpotent. If $K > 1$, let A be a nonidentity characteristic abelian subgroup of K. Then $A \lhd G$, so by Step 4, $A \leqslant \mathbf{Z}(G)$. But by Dornhoff [1], $K \cap \mathbf{Z}(G) = 1$, a contradiction.

Step 6. *N is cyclic, $\chi(1) = 1$, and so χ is monomial.*
By Rigby [7], the odd Sylow subgroups of N are all cyclic, since they are modular, while the Sylow 2-subgroup of N is cyclic since it is quaternion-free. Hence N is the direct product of cyclic groups, and is itself cyclic. Now by Step 2, $\chi \mid N \in \operatorname{Irr} N$, so $\chi(1) = 1$.

THEOREM 3.3. *Let G be a finite group, with $N \lhd G$. Suppose G/N is solvable, and for each $H \geqslant N$, the H-chief sections of H/N have odd rank. Let $\chi \in \operatorname{Irr} G$. Then χ is induced over N.*

Proof. By induction on $(G : N)$. Let $\mathscr{S} = \{M \lhd G \mid N \leqslant M, \chi \mid M \notin \operatorname{Irr} M\}$. If $\mathscr{S} = \varnothing$, then $\chi \mid N \in \operatorname{Irr} N$, and χ is trivially induced over N. If $\mathscr{S} \neq \varnothing$, pick $M \in \mathscr{S}$ with $|M|$ as large as possible. Now $M \neq G$ since $\chi \in \operatorname{Irr} G$. Let $K \lhd G$ so that K/M is a chief section of G. By the maximality of $|M|$, $\chi \mid K \in \operatorname{Irr} K$. Now $|K/M| = p^d$ for some prime p and odd integer d. Inspect the three possibilities in Theorem 2.3. Since $\chi \mid M \notin \operatorname{Irr} M$, (a) does not occur. Since d is odd, (b) does not occur. Hence

$$\chi \mid M = \sum_{i=1}^{p^d} \theta_i, \qquad \theta_i \in \operatorname{Irr} M.$$

Let $H = \mathbf{I}_G(\theta_1)$. By Clifford's theorem, there is $\xi \in \operatorname{Irr} H$, such that $\xi^G = \chi$, $\xi \mid M = \theta_1$, and $(G : H) = p^d > 1$. By induction ξ is induced over N, so $\chi = \xi^G$ is induced over N.

COROLLARY 3.4. *Let G be a solvable group. Let $N \lhd G$. Suppose*

(i) *for each $N \leqslant H \leqslant G$, all chief sections of H/N have odd rank;*

(ii) *the odd Sylow subgroups of N are all modular, and the Sylow 2-subgroups of N are quaternion-free.*

Then G is an M-group.

THEOREM 3.5. *Let G be a finite group, $N \lhd G$. Let $N \leqslant M \lhd G$ with $(G : M)$ and $(M : N)$ relatively prime. Let $\chi \in \operatorname{Irr} G$, and let θ be an irreducible constituent of $\chi \mid M$. Suppose χ is induced over N. Then χ is induced over M and θ is induced over N.*

Proof. Let $H \leqslant G$ with $\xi \in \operatorname{Irr} H$ induce χ over N. Then ξ^{HM} is an irreducible constituent of $\chi \mid HM$. Hence $\xi^{HM} \mid M$ is a sum of G-conjugates of θ. Thus, replacing H and ξ by conjugates if necessary, we may assume that θ is a constituent of $(\xi^{HM}) \mid M$. By Lemma 2.1, $(\xi^{HM}) \mid M = (\xi \mid (H \cap M))^M$, and by Frobenius reciprocity, $\xi \mid (H \cap M)$ is a constituent of $\theta \mid (H \cap M)$. Hence, $\xi \mid N$ is an irreducible constituent of $\theta \mid N$.

Let π be the set of primes dividing $(M : N)$. Now, $\chi(1)/\xi(1) = (G : H)$. So

$$(\chi(1)/\xi(1))_\pi = (G : H)_\pi = (HM : H).$$

Also, $\chi(1)/\theta(1)$ divides $(G : M)$ [2, p. 570]. Thus $\chi(1)/\theta(1)$ is a π'-number. Similarly, $\theta(1)/\xi(1)$ is a π-number. But

$$\chi(1)/\xi(1) = \{\chi(1)/\theta(1)\}\{\theta(1)/\xi(1)\}.$$

Hence

$$\theta(1)/\xi(1) = \{\chi(1)/\xi(1)\}_\pi = (HM : H) = (M : H \cap M).$$

Now $(\xi \mid (H \cap M))^M$ has θ as a constituent, and

$$(\xi \mid (H \cap M))^M(1) = (M : H \cap M)\,\xi(1) = \theta(1).$$

Thus $(\xi \mid (H \cap M))^M = \theta$, and of course $(\xi \mid (H \cap M)) \mid N = \xi \mid N \in \operatorname{Irr} N$. Thus the character $\xi \mid (H \cap M)$ of the subgroup $H \cap M$ induces θ over N. Similarly, since $(\xi^{HM}) \mid M = (\xi \mid (H \cap M))^M$ is an irreducible character of M, the character ξ^{HM} of the subgroup HM induces χ over M.

Dade has given a method of embedding any solvable group in an M-group [2, Theorem V. 18.10]. The next lemma and theorem generalize his method to the situation studied in this section. Let G be a group, and H a permutation group on the symbol set Ω. The wreath-product $G \wr H$ of G with H is the set

$$\{(f, h) \mid h \in H, f \text{ mapping } \Omega \text{ into } G\},$$

together with the multiplication

$$(f_1, h_1)(f_2, h_2) = (g, h_1h_2)$$

with $g(i) = f_1(i)\,f_2(i^{h_1})$ for $i \in \Omega$.

LEMMA 3.6. *Let H be a finite group, $N \lhd H$, and suppose, for every $\chi \in \operatorname{Irr} H$, that χ is induced over N. Let Z be a cyclic group of prime order p, acting as a right regular permutation group on itself. Let $G = H \wr Z$. Let $N_1 = \{(f, 1) \mid f : Z \to N\}$. If $\theta \in \operatorname{Irr} G$, then θ is induced over N_1.*

The proof is similar to the one of Dade mentioned above.

THEOREM 3.7. *Let G be a finite group, $N \lhd G$ with G/N solvable. Then there is a finite group G_1 with normal subgroup N_1 such that*

(i) $G \leqslant G_1$.

(ii) $G \cap N_1 = N$.

(iii) *Every* $\chi \in \operatorname{Irr} G_1$ *is induced over* N_1.

4. Inductive Groups

Let K be a finite group. Call K *inductive* if, whenever G is a finite group and α is a homomorphism of G onto K, then every irreducible character of G is induced over Ker α. In Theorem 3.3, we showed that if every chief section of every subgroup of the solvable group K has odd rank, then K is inductive. Theorem 4.1 gives two characterizations of inductive groups.

Let K be an inductive group. Every irreducible character of K is induced over the identity subgroup of K, and so K is an M-group. In particular, K is solvable.

THEOREM 4.1. *Let K be a finite solvable group. The following are equivalent.*

(a) *K is inductive.*

(b) *Every subgroup of K has a covering group (for projective representations) which is an M-group.*

(c) *No subgroup of K has a symplectic chief section. (A chief section of a solvable group is called sympletic if there is a nonsingular symplectic form on the section which is invariant under the inner automorphisms of the group.)*

To illustrate the kind of M-group that is not inductive, let $K = A_4$, the alternating group on four letters. By Corollary 3.4, it is an M-group. Let $G = SL(2, 3)$. The faithful irreducible character χ of G has degree 2, and cannot be induced over $\mathbf{Z}(G)$, since $G/\mathbf{Z}(G) \simeq K$ has no subgroup of index 2. So K is not inductive. Also, G is a covering group of K and χ is not monomial, so G is not an M-group, and K does not satisfy condition (b). Finally, the commutator map from $\mathbf{O}_2(G)$ to $\mathbf{Z}(G)$ induces a symplectic form on the minimal normal subgroup $\mathbf{O}_2(K)$ of K, and K does not satisfy condition (c).

We now begin the proof of Theorem 4.1, and show that (c) implies something stronger than (b). Assume (c). Let H be a subgroup of K. Let F be any finite central extension by H (not just a covering group for H), so there is $C \leqslant \mathbf{Z}(F)$ with $F/C \simeq H$. If F is not an M-group, then some chief section A/B of some subgroup L of F is ramified [6, Theorem 1.2]. That means there is $\theta \in \operatorname{Irr} B$ which is invariant in L and θ does not extend to an irreducible character of A. Thus, A/B admits a nonsingular L-invariant symplectic form [5, Theorem 2.7]. Define α mapping $L/\mathbf{C}_L(A/B)$ onto $LC/\mathbf{C}_{LC}(AC/BC)$ by $(g\mathbf{C}_L(A/B))\alpha = g\mathbf{C}_{LC}(AC/BC)$. Since $C \leqslant \mathbf{Z}(F)$ and $A \nleqslant \mathbf{Z}(F)$, the map α is an isomorphism. Now A/B and AC/BC are isomorphic LC-sections. Thus LC has the symplectic section AC/BC, and then LC/C has the symplectic section $\{AC/C\}/\{BC/C\}$. But LC/C is isomorphic to a subgroup of K, contradicting (c).

We now prepare for the proof that (b) implies (a).

PROPOSITION 4.2. *Suppose that the finite solvable group H has a covering group H^* (for projective representations) which is an M-group. Let F be a finite extension by H, that is, there is a homomorphism α of the finite group F onto H. Let $N = \operatorname{Ker} \alpha$. Suppose $\chi \in \operatorname{Irr} F$ and $\chi \mid N = e\theta$ for some integer e and $\theta \in \operatorname{Irr} N$. Then χ is induced over N.*

Proof. Since H^* is a covering group for H, there is a central subgroup C of H^*, and a mapping ρ of H into H^* so that $h \to Ch^\rho$ is an isomorphism of H onto H^*/C. By Clifford's theorem [2, p. 567] and the properties of covering groups [2, p. 639], there is an ordinary matrix representation X of H^* of degree e and projective matrix representation Y of F of degree $\theta(1)$ such that

$$T(f) = X(f^{\alpha\rho}) \otimes Y(f) \qquad (f \in F),$$

is an ordinary matrix representation of F affording χ. (The symbol $\otimes$ denotes the Kronecker product of the matrices.)

By assumption H^* is an M-group. So there is a subgroup L^* of H^* of index e for which $X \mid L^*$ has a linear constituent λ. Since C is represented by scalar matrices, $C \leqslant L^*$. Because $X \mid L^*$ is completely reducible, we may conjugate the matrix representation X (if necessary), and assume that the matrices for $X \mid L^*$ have λ in the (1, 1)-position, and all other entries in the first row and first column of $X \mid L^*$ are zero. Let $L = \{f \in F \mid f^{\alpha\rho} \in L^*\}$. Let $S(g) = \lambda(g^{\alpha\rho})\, Y(g)$ for $g \in L$. From the shape of $T(g)$, we see that S is a constituent of $T \mid L$ and S is an ordinary matrix representation of L. Let ϕ be the character of L afforded by S. By construction ϕ is a constituent of $\chi \mid L$. By Frobenius reciprocity, χ is a constituent of ϕ^F. But $\phi^F(1) = (F:L)\,\phi(1) = (H^*:L^*)$ (degree Y) $= e\theta(1) = \chi(1)$. Hence $\phi^F = \chi$. Since ϕ is a constituent of $\chi \mid L$, we have $\phi \mid N$ is a constituent of $\chi \mid N = e\theta$. But $\phi(1) = \theta(1)$. Hence $\phi \mid N = \theta$ and ϕ induces χ over N.

We now show that (b) implies (a) in Theorem 4.1. Assume (b). Let G be a finite group with normal subgroup N for which $G/N \simeq K$. Let $\chi \in \operatorname{Irr} G$. Then $\chi \mid N = e\,(\theta_1 + \cdots + \theta_t)$, where e is a positive integer, and $\theta_1, \ldots, \theta_t$ are distinct irreducible characters of N. Let $T = \mathbf{I}_G(\theta_1)$, the inertia group of θ_1. Then there is a unique $\xi \in \operatorname{Irr} T$ for which $\xi^G = \chi$ and $\xi \mid N = e\theta_1$. The group T with character ξ and normal subgroup N satisfy the hypothesis of Proposition 4.2. So there is a subgroup L of T and $\phi \in \operatorname{Irr} L$ for which $\phi^T = \xi$ and $\phi \mid N = \theta_1$. Thus $\phi^G = (\phi^T)^G = (\xi)^G = \chi$ and ϕ induces χ over N.

We finally show that an inductive group satisfies condition (c) of Theorem 4.1. Let K be an inductive group. Suppose, on the contrary, that there is a subgroup of K with a symplectic chief section. Take $H \leqslant K$ of least possible order among the subgroups with a symplectic chief section. Among the symplectic chief sections of H, choose one, M/L, such that M has the largest possible order. Then M/L is an elementary abelian p-group for some prime p. We fix these choices and notation for the rest of the argument. We shall eventually show that K is not inductive, a contradiction.

Let $C = \mathbf{C}_H(M/L)$. Let $D = \mathbf{O}_{p'}$ $(H \bmod C)$. Since H/C has a nontrivial faithful irreducible module in characteristic p, $\mathbf{O}_p$ $(H \bmod C) = C$. Since H is solvable, $D \neq C$. Let S be a Hall p'-subgroup of D. Let $A = L\mathbf{N}_H(S)$. By the Frattini argument, $H = D\mathbf{N}_H(S) = CA$. Since $M \lhd H$, the product MA is a subgroup of H. Since $H = CA$, the elements of A cover the action of H on M/L. Hence M/L is a chief section of MA, and is symplectic. By the minimality of $|H|$, the subgroup MA cannot be proper, so $H = MA$.

Now consider $M \cap A$. Since M/L is abelian, M normalizes $M \cap A$. Of course A normalizes $M \cap A$. Hence $M \cap A \lhd MA = H$. Since M/L is a chief section of H, $M \cap A$ is either M or L. Suppose $M \cap A = M$, so that $M \leqslant A$. By Clifford's theorem, $M = [M, D]L = [M, S]L$. Hence,

$$M = [M, S]L \leqslant [A, S]L \leqslant SL.$$

But M/L is a p-group, while SL/L is a p'-group, a contradiction. Hence, $M \cap A = L$.

Let $C_1 = \mathbf{C}_A(M/L)$. Note that $C_1 \lhd MA = H$ and $C_1 \cap M \leqslant A \cap M = L$. Hence, C_1M/C_1 is an H-chief section of H which is H-isomorphic to M/L. By the maximality of $|M|$, we have $C_1M = M$ and $C_1 = L$. That is, $\mathbf{C}_A(M/L) = L$.

Let $H^* = H/L$, and let M^* and A^* be the images of M and A in H^*. We summarize the structure of H^*. The subgroup M^* is a minimal normal subgroup of H^*. The subgroup A^* is a complement to M^* in H^*, and is faithfully represented by conjugation as a group of symplectic automorphisms of M^*.

If p is odd, let P be the extra-special group of exponent p and order $p\,|M^*|$. If $p = 2$, let P be the central product of an extra-special group of order $2\,|M^*|$ and a cyclic group of order 4. Let U be the group of automorphisms of P which are trivial on $\mathbf{Z}(P)$. It is well known that I, the group of inner automorphisms of P, is a self-centralizing normal subgroup of U, and U/I induces (by conjugation) the full symplectic group on I. (Warning: When $p = 2$, U need not be split over I.) Since M^* and I are isomorphic symplectic vector spaces, the linear group U/I has a subgroup B/I which is isomorphic to the group A^* of symplectic automorphisms of M^*. Let $D = \mathbf{O}_{p'}$ $(B \bmod I)$. Since B/I has a faithful irreducible module in characteristic p, $\mathbf{O}_p$ $(B \bmod I) = B$. Since B is solvable, $D \neq I$. Let S be a Hall p'-subgroup of D. Let $N = \mathbf{N}_B(S)$. By the Frattini argument, $B = IN$. Since I is abelian, $I \cap N \lhd I$. Also, $I \cap N \lhd N$ and hence $I \cap N \lhd IN = B$. Since I is an irreducible B/I-module, $I \cap N$ is either 1 or I. If it is I, then $I \leqslant N = \mathbf{N}_B(S)$ and $[I, S] \leqslant I \cap S = 1$. This contradicts the fact that I is self-centralizing in U. Hence $I \cap N = 1$. Thus B is the semidirect product of I and N. By our choice of B, there is an isomorphism of H^* onto B which carries M^* to I.

So we may regard H^* as a group of automorphisms of P which are trivial on $\mathbf{Z}(P)$ and in which the subgroup M^* induces the group of inner automorphisms on P.

Form the semidirect product PH^*. Then PM^* is extra-special if p is odd, and is the central product of an extra-special group with $\mathbf{Z}(P)$ if $p = 2$. Let θ be a faithful irreducible character of PM^*. Then θ vanishes off of $\mathbf{Z}(P)$ and thus θ is invariant in PH^*. Let λ be the unique irreducible constituent of $\theta \mid \mathbf{Z}(P)$. Now $\mathbf{Z}(P)H^* = \mathbf{Z}(P) \times H^*$ so consider the character $\mu = \lambda \# 1_{H^*}$ of $\mathbf{Z}(P)H^*$. Let χ be an irreducible constituent of μ^{PH^*}. Then μ is a constituent of $\chi \mid \mathbf{Z}(P)H^*$ and hence λ is an irreducible constituent of $\chi \mid \mathbf{Z}(P)$. Since θ is the only element of Irr PM^* which contains λ when restricted to $\mathbf{Z}(P)$, it is the only irreducible constituent of $\chi \mid PM^*$. Hence

$$\theta(1) \leqslant \chi(1) \leqslant \mu^{PH^*}(1) = (PH^* : \mathbf{Z}(P)H^*) = (P : \mathbf{Z}(P)) = \theta(1).$$

Thus $\theta(1) = \chi(1)$ and χ is an extension of θ to PH^*.

We next argue that χ is not induced over P. Now $\chi \mid P = \mid M^* \mid^{1/2}\phi$ for some $\phi \in \text{Irr } P$. If χ is induced over P, then there is $J \leqslant PH^*$ with $(PH^* : J) = \mid M^* \mid$. By Lemma 2.2, $PH^* = J(PM^*)$. Hence, $(PM^* : J \cap PM^*) = \mid M^* \mid^{1/2}$. Since PM^*/P is abelian, $J \cap PM^* \lhd PM^*$. This contradicts the fact that PM^*/P is a chief section of PH^*.

Now form the semidirect product PH by using the natural homomorphism from H to H^* to define an action of H on P. Regard χ as a character of PH having L in its kernel. Let T be a right transversal to H in G which contains 1. Let $F = \{(\alpha, g) \mid \alpha : T \to P, g \in G\}$. For (α_1, g_1) and (α_2, g_2) in F, define their product to be $(\beta, g_1 g_2)$, where

$$\beta(t) = \alpha_1(t)\, \alpha_2(t')^{h'^{-1}}$$

for $t \in T$ and $tg_1 = h't'$ with $h' \in H$, $t' \in T$. Then F is the twisted wreath product $P \wr_H G$ [2, pp. 99–100].

Let $D = \{(\alpha, 1) \mid \alpha : T \to P, 1 \in G\}$. Then D is isomorphic to a direct product of $\mid T \mid = (G : H)$ copies of P. D is clearly normal in F. Also, $F/D \simeq G$. Let $D_1 = \{(\alpha, 1) \mid (\alpha, 1) \in D,\ \alpha(1) = 1\}$. Let $S = \{(\alpha, h) \mid \alpha : T \to P,\ \alpha(t) = 1$ if $t \neq 1, h \in H\}$. Then $S \simeq PH$. Also, $D_1 \cap S = 1$ and S normalizes D_1. Regard χ as a character of D_1S having D_1 in its kernel. Now $\chi \mid D$ is homogeneous; let ξ be the irreducible constituent of $\chi \mid D$. Then $D_1 \leqslant \text{Ker } \xi$ but $D \cap S \nleqslant \text{Ker } \xi$. Now, if $f \in F - D_1S$, then $f^{-1}(D \cap S)f \leqslant D_1$. Hence, $\xi^f \neq \xi$. Thus, the inertia group of ξ lies in D_1S, so it must be D_1S. Let $\psi = \chi^F$. Let β be an irreducible constituent of ψ. By Frobenius reciprocity, ξ is a constituent of $\beta \mid D$. By Clifford's theorem, there is a unique $\zeta \in \text{Irr}(\mathbf{I}_L(\xi))$ for which both $\zeta \mid D$ has ξ as a constituent, and β is a constituent of ζ^F. Furthermore, $\zeta^F = \beta$. But χ satisfies the conditions which characterize ζ. Hence, $\chi = \zeta$ and $\chi^F = \beta$. Thus, $\chi^F = \psi$ is irreducible.

Now, G is an inductive group. So there is a subgroup J of F with $D \leqslant J$, and $\alpha \in \operatorname{Irr} J$ such that $\alpha^F = \psi$ and $\alpha \mid D = \xi$. Hence $J \leqslant \mathbf{I}_L(\xi) = D_1S$. Consider α^{D_1S}. By the characterization of ζ in the previous paragraph, $\alpha^{D_1S} = \zeta = \chi$. Hence, χ is induced over D. But $D_1 \leqslant \operatorname{Ker} \chi$. So χ, regarded as a character of S, is induced over $D \cap S$. But $S \simeq PH$ and $D \cap S \simeq P$, contradicting the construction of χ as a character of PH^* which was not induced over P. This contradiction shows that no subgroup of an inductive group has a symplectic chief section, and completes the proof of Theorem 4.1.

5. Formations of Inductive Groups

We now consider the class $\mathscr{F}$ of inductive groups. It is a formation (Theorem 5.1), but not a saturated formation (Proposition 5.2). The subformation $\mathscr{O}$ of inductive groups of odd order is saturated, and we give a local definition, and an alternative description of $\mathscr{O}$ (Theorem 5.3).

To show that a given group G is inductive, we employ Theorem 4.1, and show that no subgroup H of G has a symplectic chief section. We often argue that if H has a symplectic chief section, then some other group, which we are assuming to be inductive, also has a subgroup with a symplectic chief section. This contradicts Theorem 4.1.

THEOREM 5.1. *Let $\mathscr{F}$ be the class of inductive groups.*

(a) *If $G \in \mathscr{F}$ and H is a subgroup of G, then $H \in \mathscr{F}$.*
(b) *If $G \in \mathscr{F}$ and H is a homomorphic image of G, then $H \in \mathscr{F}$.*
(c) *If G_1 and G_2 are in $\mathscr{F}$, then $G_1 \times G_2$ is in $\mathscr{F}$.*
(d) *$\mathscr{F}$ is a variety and a formation.*
(e) *Let G be a finite group with $G/\mathbf{Z}(G) \in \mathscr{F}$. Then $G \in \mathscr{F}$.*

Proof. (a) By Theorem 4.1, it is sufficient to show that no subgroup K of H has a symplectic chief section. But if K has a symplectic chief section, then K is a subgroup of G with a symplectic chief section, contradicting Theorem 4.1 for G.

(b) Similar to (a).

(c) By Theorem 4.1, it is sufficient to show that no subgroup H of $G_1 \times G_2$ has a symplectic chief section. Since G_1 and G_2 centralize each other in $G_1 \times G_2$, every chief section of H is H-isomorphic to an H-chief section of $H/(H \cap G_1)$ or an H-chief section of $H/(H \cap G_2)$. But $H/(H \cap G_1)$ (resp. $H/(H \cap G_2)$) is isomorphic to a subgroup of G_2 (resp. G_1) and, by Theorem 4.1, has no symplectic chief section.

(d) This follows immediately from parts (a), (b), and (c) and the definition of a variety and a formation.

(e) It suffices to show that no subgroup H of G has a symplectic chief section. By the Jordan–Hölder theorem, every chief section of H is H-isomorphic to one obtained in a chief series which refines the normal series $1 \leqslant H \cap \mathbf{Z}(G) \leqslant H$. Since $H \cap \mathbf{Z}(G) \leqslant \mathbf{Z}(H)$, no chief section of H below $H \cap \mathbf{Z}(G)$ is symplectic. A chief section of H above $H \cap \mathbf{Z}(G)$ is (trivially) isomorphic to a section of $H/(H \cap \mathbf{Z}(G))$ which is H-isomorphic to $H\mathbf{Z}(G)/\mathbf{Z}(G)$, a subgroup of $G/\mathbf{Z}(G)$. By Theorem 4.1, $H\mathbf{Z}(G)/\mathbf{Z}(G)$ does not have a symplectic chief section.

Proposition 5.2. *$\mathscr{F}$ is not a saturated formation.*

Proof. Let p be a prime congruent to 1 mod 24, for example, take $p = 73$. Let F be the field of p elements. Let $P = \{(u_1, u_2, u_3, u_4, u_5, u_6) \mid u_i \in F\}$ and define a multiplication of P by setting

$$
\begin{aligned}
&(u_1, u_2, u_3, u_4, u_5, u_6)(v_1, v_2, v_3, v_4, v_5, v_6)\\
&\qquad = (u_1 + v_1, u_2 + v_2, u_3 + v_3 + u_1 v_2, u_4 + v_4, u_5 + v_5, u_6 + v_6 + u_4 v_5).
\end{aligned}
$$

It is easy to verify that P is a group of order p^6 and exponent p. The Frattini subgroup $\Phi(P)$ is $\{(0, 0, u_3, 0, 0, u_6) \mid u_3, u_6 \in F\}$.

Let $H = \langle x, t \mid x^3 = t^8 = 1, t^{-1}xt = x^{-1}\rangle$, a group of order 24. Fix a primitive third root ω and a primitive fourth root ρ of unity in F. Form the semidirect product $G = PH$ by setting

$$x^{-1}(u_1, u_2, u_3, u_4, u_5, u_6)x = (\omega u_1, \omega u_2, \omega^2 u_3, \omega^2 u_4, \omega^2 u_5, \omega u_6)$$

and

$$t^{-1}(u_1, u_2, u_3, u_4, u_5, u_6)t = (u_4, u_5, u_6, \rho u_1, \rho u_2, -u_3),$$

for $(u_1, u_2, u_3, u_4, u_5, u_6) \in P$. It is easy to verify that these definitions extend to an action of H on P. Note that $\Phi(P) \lhd G$, then that $\Phi(P) = \Phi(G)$.

We now show that $G/\Phi(G)$ is inductive. Since $p \equiv 1 \pmod{24}$, any noncyclic chief section of a subgroup of $G/\Phi(G)$ must be in fact H-invariant, and be a subquotient V of $P/\Phi(P)$. Switch to additive notation on V. The element t induces a linear transformation on V, and the determinant of the action of t is $-\rho$. Since all symplectic transformations have determinant 1, V is not a symplectic chief section. By Theorem 4.1, $G/\Phi(G)$ is inductive.

Now consider the minimal normal subgroup $\Phi(G)$ of G. In additive notation, it is a two-dimensional vector space over F. The elements x and t induce linear transformations on $\Phi(G)$ of determinant 1. Since the symplectic and special linear groups coincide in dimension 2, $\Phi(G)$ is a symplectic chief section of G, and by Theorem 4.1, G is not inductive. This group G shows that $\mathscr{F}$ is not saturated.

The example constructed in the previous proof depended on special properties of the prime 2. We now consider the subformation $\mathscr{O}$ of inductive groups of odd order. Schacher and Seitz [9] considered the formation $\mathscr{I}$ of (solvable) groups of odd order in which every chief section of every subgroup has odd rank (compare Theorem 3.3). As is alluded to in the last paragraph of [9], the formations $\mathscr{I}$ and $\mathscr{O}$ are identical.

We define the local formations which will be used to induce $\mathscr{O}$. $\mathscr{O}(2)$ is the empty formation. For p an odd prime, $\mathscr{O}(p)$ is the formation of (solvable) groups H of odd order such that for every prime divisor $q \in \pi(H) - \{p\}$, the prime p has odd order in the multiplicative group of Z_q, the field of integers modulo q.

THEOREM 5.3. *The formations $\mathscr{O}$ and $\mathscr{I}$ are identical. The formation $\mathscr{O}$ is locally induced by the $\mathscr{O}(p)$.*

Proof. Let $\mathscr{L}$ be the formation locally induced by the $\mathscr{O}(p)$. We show $\mathscr{I} \leqslant \mathscr{O} \leqslant \mathscr{L} \leqslant \mathscr{I}$.

(1) $\mathscr{I} \leqslant \mathscr{O}$. This has been shown in Theorem 3.3.

(2) $\mathscr{O} \leqslant \mathscr{L}$. Let $G \in \mathscr{O}$ and K/L be a chief section of G. Then K/L is an elementary abelian p-group for some odd prime p. Let $C = \mathbf{C}_G(K/L)$ and $A = G/C$. Let $q \in \pi(A) - \{p\}$ Take $x \in G$ so that xC has order q in A Since x does not centralize K/L, some $\langle x \rangle$-invariant irreducible subquotient K_0/L_0 of K/L is not centralized by x. By Theorem 4.1 applied to the subgroup $\langle x \rangle C$ of G, K_0/L_0 is not a symplectic $\langle x \rangle$-module. By [6, Lemma 2.2(a)], p has odd order in Z_q. Thus $G \in \mathscr{L}$.

(3) $\mathscr{L} \leqslant \mathscr{I}$. Let $G \in \mathscr{L}$. Since every subgroup of a group in $\mathscr{L}$ is again in $\mathscr{L}$, we may assume, by induction on $|G|$, that every proper subgroup of G has all chief sections of odd rank. By [9, Lemma 4], every chief section of G has odd rank. Thus $G \in \mathscr{I}$.

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